

Simple alternative to disjunctive kriging

I. Clark M.Sc., Ph.D., D.I.C., C.Eng., F.S.S., F.I.M.A., M.I.M.M.

Department of Mineral Resources Engineering, Imperial College of Science and Technology, London

F. L. Clausen M.Sc.

Mineralogical Institute, Technical University of Denmark, Copenhagen, Denmark

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Synopsis

The regression effect in ore-reserve estimation has been identified, described and quantified in vein and reef deposits for more than twenty years. Its importance in 'three-dimensional' deposits - whether underground or open-pit -- has been recognized under the alternate guise of the geostatistical volume-variance relationship. The Fontainebleau school has developed the technique of disjunctive kriging to combat the regression-volume-variance effect, but this is not yet widely available (or understood) in a form in which the ordinary user could apply it. A new and simpler approach to this problem is proposed, its development on a Cornish tin vein is briefly described and a fuller case study of its application to part of a three-dimensional uranium deposit in South Greenland is given.

There are many possible sources for the difference between an ore-reserve estimate and the actual amount of ore in a stope or mining block, some of which present insurmountable problems in that they reflect the limited amount of sampling available compared with the large volume of ground to be estimated. They also, often, reflect the peripheral nature of the sampling and/or the erratic nature of the mineral that is sampled. Many attempts have been made to minimize the errors incurred in estimation, and the current consensus of opinion seems to have focused on the weighted-average type of estimator. Each sample is given a weight in accordance with its geometrical position in relation to the volume that is to be estimated. The weights may be decided by one of several methods: simple averaging, inverse distance and related operators or kriging. A weighted average or *linear* type of estimator has many advantages in practice: it is easy to compute and intuitively pleasing, and, if the user can derive a semivariogram for the deposit, the standard error of such an estimate may be evaluated without difficulty.

Unfortunately, there is one major drawback in the use of a linear combination of sample values to estimate the average value over a large volume of ground. First identified in the South African gold mines by Krige in 1951¹, the problem may be stated as follows: high-grade blocks are relatively overvalued and low-grade blocks are relatively undervalued. Krige suggested that this 'bias' in the estimation process could be eliminated by the production of a regression curve that would relate the linear estimator to a more accurate 'unbiased' estimator. With the use of stope sampling from worked-out areas and by the

comparison of these with linear estimates obtained from development data, an empirical curve could be produced which would then be used to 'correct' the linear estimators. Because of Krige's pioneering work in the field, the bias on the estimator became known as the *regression effect*.

The topic appears frequently in the literature, with papers by Krige^{2,3}, Pryor and co-workers⁴, Royle and Newton⁵ David and co-workers⁶, Marchal⁷ and many others. After the introduction of Matheron's theory of geostatistics in the early 1960s the problem arose under another name: the 'volume-variance' relationship. Matheron showed⁸ that the simple explanation of the 'bias' on the estimators, and hence on the grade-tonnage curves, was due to the fact that the *estimator*, a weighted average of relatively small samples, has a different *variance* from the average of the stope or block to be estimated. From this an 'unbiased' grade-tonnage curve may be produced, if the user is willing to assume or derive a probability distribution for the sample values within the deposit. This method, however, can only correct the bias on the global grade-tonnage curve, not on the individual stope or block estimates - that is, the error on the global estimate may be quantified, but the errors on each local estimation cannot.

The situation may, thus, be summarized as follows. The regression effect may be quantified if samples are available in worked-out areas. This allows future estimates to be corrected for the so-called 'bias'. The volume-variance relationship may be evaluated before any stope samples are available; however, it can only correct the global estimation.

To remedy the shortcomings in the geostatistical approach, Matheron⁸ dealt with non-linear estimators and introduced the theory of a new technique called 'disjunctive kriging'. Journel and Huijbregts⁹ did not consider that non-linear kriging could be counted among the 'well tried and proven techniques', and, although reports have appeared of the relative merits of disjunctive kriging as an estimation method, and various short (and long) courses are being run, it is still difficult for the average ore-reserve analyst either to practice or to assess the technique. One major practical objection to the use of disjunctive kriging is that it is highly dependent on the histogram of values obtained from the samples. In theory, the probability distribution of the samples must be transformed to a standard normal (Gaussian) distribution. In practice, however, the histogram of the samples appears to be taken as absolutely representative of the complete deposit. It is obvious that, unless the samples have been taken 'randomly and independently' throughout the deposit, this is unlikely to be accurate. Samples are generally taken according to some systematic plan, with more samples available in areas of interest and so on. A technique that depends on which set of sample values is considered does not seem to promise a great deal of robustness in practice. Must all block estimates be changed every time a new sample is taken?

There are other problems with the application of disjunctive kriging that are just as fundamental but may not be so obvious to the uninitiated user. It is possible, for instance, when the transformation to 'standard normality' is used, to arrive at a probability function that takes negative values. Then there is the question of reversion of estimates made on 'standardized' values back to values that make sense in terms of grade. Gross errors in predicted grade can be produced on the back-transformation and are not always identifiable until after mining has taken place.

The purpose of this paper, however, is not to denigrate disjunctive kriging as such but rather to present a simple technique, which appears to correct for the 'bias' on grade-tonnage calculations, without the necessity for involved and possibly unstable mathematics.

Clark¹⁰ was able to carry out a detailed appraisal of several methods of ore-reserve estimation from a large data set from a partially worked tin lode in Cornwall. Many conclusions were derived from that study, some of which have been published elsewhere¹¹. It was found that Krige's regression approach to stope estimation increased the accuracy of prediction markedly, although it could not be used with confidence until a fairly large area had already been stoped. It was also found that semi-variograms could be constructed and modelled on this deposit and that kriging could be seen to work on this lode. Unfortunately, kriging estimates for the stopes turned out to be identical to the arithmetic mean already used at the mine. These had already been proved to be worse than the empirical regression estimators. Kriging estimates (and arithmetic means), however, can be produced before stope-sample information is available by the use of only development data. Because of the complex mineralization process within the lode, it was found that the distribution of sample values had to be characterized by a mixture of two log-normal distributions¹². Because of this, there was no simple transformation that would produce values that followed the standard normal distribution. Given these circumstances, the senior author attempted to unite the two approaches to regression-volume-variance and evolve a regression estimator before stoping information was available. For brevity, this method is referred to here as 'georegression'.

Derivation and testing of georegression

A full technical and mathematical derivation of georegression was given by Clark¹³; however, a brief description of the derivation of the method and some of the problems encountered in its application is more appropriate here. The study¹³ was carried out on the Simms Lode at Geevor Tin Mines, Ltd., in Cornwall, where sampling and estimation problems are similar to those in the South African reefs, although Simms Lode is rather narrow and almost vertical. A limited amount of peripheral sampling is available for use in the estimation of the internal average of a rectangular-stoping block. In this case, panels, which are 125 ft (38.1 m) long, must be estimated from densely sampled development drives - 100 ft (30.5 m) apart. It can be shown that weighted averages such as kriging are no better than the use of the arithmetic mean in such situations. In March, 1972, most development on the central part of the lode had been completed (> 2400 samples), one level was completely stoped out and three were at various stages of stoping (~1600 samples). By March, 1976, all four of these levels had been stoped out and two new stopes started, which gave ~2700 stope samples in all. This was an ideal case for evaluation of the classical regression analysis on the 1972 data and then testing on the 1976 data. Where stope sampling was available, panels approximately 125 ft x 100 ft (38.1 m x 30.5 m) were marked out and the arithmetic mean of the internal samples was calculated. This was taken as an adequate representation of the value of ore contained within the panel. No attempt was made to allow for the log-normal characteristics of the sample distribution, partly because of the complex nature of the model and partly because it was desired to see whether a technique could be developed that would be independent of the grade distribution. For each of these panels the corresponding development average was calculated, i.e. the arithmetic mean of all drive samples peripheral to the panel.

Fig. 1 shows a graph of the development averages plotted against the corresponding stope averages. Also shown are the 45° line, which would indicate that the development average was the 'best' estimator, and the classical least-squares regression line. The

difference between the arithmetic mean (or kriged value) and the regression relationship is immediately obvious, and it can be shown statistically that the regression line is a significantly better fit to the data.

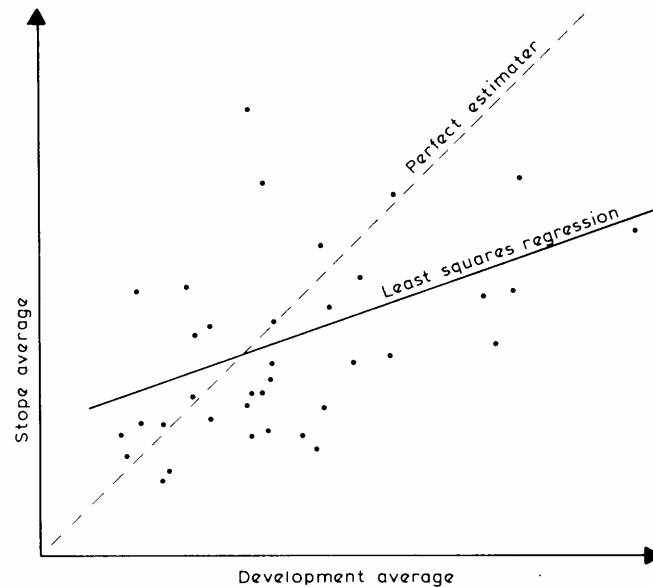


Fig. 1 Comparison of estimated and actual values of Simms Lode (west of fault) for 125-ft panels

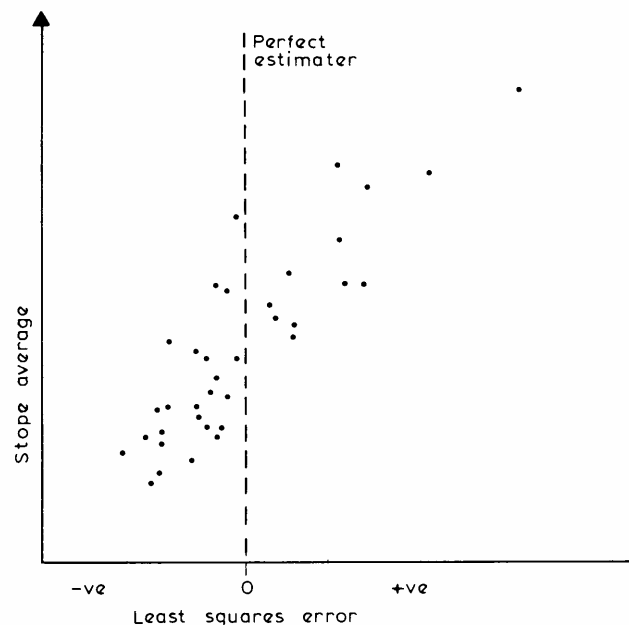


Fig. 2 Estimation error against actual values, from least-squares regression on Simms Lode (WOTF)

If the regression estimator is adopted, the main variable of interest is the error that is incurred in the prediction. This may easily be calculated for the members of the data set, since it is simply the difference between the actual stope average and the calculated regression value. For more information about the behaviour of such an estimator this quantity was plotted against the stope average; it is shown in Fig. 2. The graph is rather disturbing since it shows that the regression error is almost perfectly correlated to the stope average. The least-squares regression line seems to have massively overcompensated for the estimation bias. Instead of overestimating high values, it considerably underestimates

them. Similarly, low values are now grossly overestimated. This could lead to large numbers of unpay panels being included in the reserves, and the expected value of the high-grade blocks is depressed.

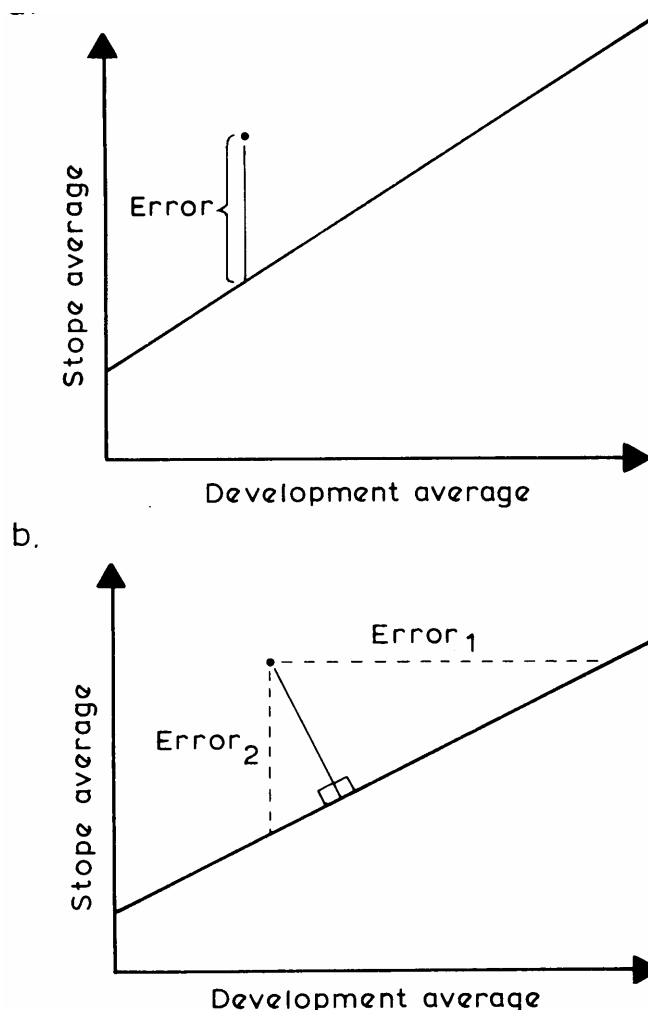


Fig. 3 Possible error criteria for regression analysis: (a) least squares; and (b) perpendicular distance

Least-squares regression assumes that the known value, or 'independent' variable, is accurately measured and that the unknown or 'dependent' variable shows only random error about the prediction line. In this particular case the stope average is itself only an estimate of the true stope/panel value, so errors exist in both 'directions'. This problem is illustrated in Fig. 3, which shows the usual assumption for least-squares analysis; the error is shown to be the vertical distance to the regression line. The set of such errors calculated from the data set is minimized to produce the coefficients for the regression line. Fig. 3(b) shows an alternative approach, which considers that both variables may be prone to error. In this case it is the perpendicular distance to the line that is minimized. It was found from the use of this criterion that a regression line could be produced that lay between the arithmetic mean and the least-squares regression and showed good predictive power when used on the 1972-76 stoping data.

The coefficients of the perpendicular-distance regression line are given by the expressions

$$\hat{b} = q \pm \sqrt{(1 + q^2)}$$

where

$$q = (s_1^2 - s_2^2) / (2 s_{12})$$

and

$$\hat{a} = (\text{overall stope mean}) - (\hat{b} \times (\text{overall development mean}))$$

where s_1 , is the standard deviation of stope averages; s_2 , the standard deviation of development averages; s_{12} , the covariance between stope and development averages; $\hat{b}$, the slope of the regression line; and $\hat{a}$, the intercept of the regression line. For comparison, in the method of least-squares regression $\hat{b}$ is s_{12}/s_2^2 .

The problem is to derive the regression relationship without previous knowledge of the stope averages. If a semi-variogram can be calculated for the deposit, this is not at all difficult. The estimator that is set up is no longer a simple linear combination of the sample values but a linear combination plus an additive constant, a . The process for the evaluation of the estimation variance of such an estimator, and hence its standard error, is broadly similar to that used for the usual weighted-average estimator. The main difference between this process and the usual one is that, in the case of the regression estimator, it must be assumed that the overall mean of the deposit is known; otherwise, this overall mean must be estimated. If the overall mean is known, it can be shown that the regression estimator, which gives the lowest estimation variance, is identical to that given by least squares. From the geostatistical terminology, rather than the classical one, the terms to be calculated are:

- (1) s_2^2 , the variance between the values of the ‘estimators’ – in this case, the averages of sets of drive samples: this is normally written as $C - \bar{\gamma}(\text{samples}, \text{samples})$;
- (2) s_{12} , the covariance between the linear estimator and the panel that is to be estimated: this is normally written as $C - \bar{\gamma}(\text{panel}, \text{samples})$.

Similarly, the other term in the calculation of a perpendicular distance regression line could be described as: s_1^2 , the variance between the true average values of the panels being estimated: that is, $C - \bar{\gamma}(\text{panel}, \text{panel})$.

All three of these $\bar{\gamma}$ terms would be calculated through the volume-variance relationship. A clear and detailed explanation of such calculations has been given by Parker¹⁴ and need not be repeated here. One problem remains: that of the assumption or estimation of the overall mean of the deposit (or the portion that is under consideration). In the classical regression approach this is assumed to be the overall mean of the areas that are worked out. With the geostatistical approach, which lacks production data, this obviously cannot be done. It is necessary, therefore, to estimate the mean of the study area before georegression may be carried out. The estimated mean value may be denoted by m , and its associated standard error by σ_m . The quantity m is then used in the calculation of the coefficient a of the regression line.

In the expressions given above, C is the value taken by the sill of the semi-variogram model. The variances and covariances cannot be evaluated if the model has no sill, so the

proposed method can only be used with certain models. Since this is also true of the volume-variance approach in general, the constraint is not as severe as it seems. It is also fairly common practice to introduce a 'false' sill into models such as the linear, which have no natural sill; the validity of this practice has not been established.

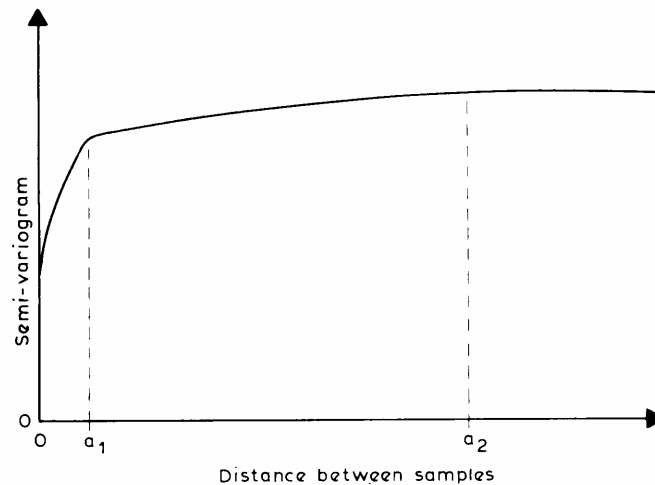


Fig. 4 Semi-variogram model for development samples only, Simms Lode (WOTF)

In the Geevor example, a model was fitted to the semi-variogram of the assay values of the development samples. This model was comprised of two spherical components and a large nugget effect. The shape of the model is shown in Fig. 4, from which two georegression lines were evaluated. The first used the principle of least-squares, since the theory predicts that this should be the most accurate. The other used the perpendicular-distance criterion. These two lines were compared with the classical-regression line, which was constructed on all of the data available in 1976, i.e. 1950 relevant development samples and 2700 stope samples. The perpendicular distance georegression, which uses only the development samples and the geometric layout of the stopping panels, compared remarkably well with the 1976 regression line. The least-squares georegression bore little resemblance to the 1976 line, contrary to the theoretical results.

One possible explanation for this was that the large nugget effect present in the model – that is, the random component – found expression not through the least squares solution but only through the perpendicular distance. With this in mind, the analysis was repeated with lode-width values from the same samples. The semivariogram for these measurements also had two spherical components, with about the same ranges of influence as the assays. It had, however, an almost negligible nugget effect. In the comparison of the 1976 regression line with the two georegressions for these values, the least-squares line was found to be a much closer approximation than the perpendicular distance. Fig. 5(a) shows the 1976 classical regression line and the perpendicular-distance georegression for the assay values. Fig. 5(b) shows the classical line and the least-squares georegression for the lode widths.

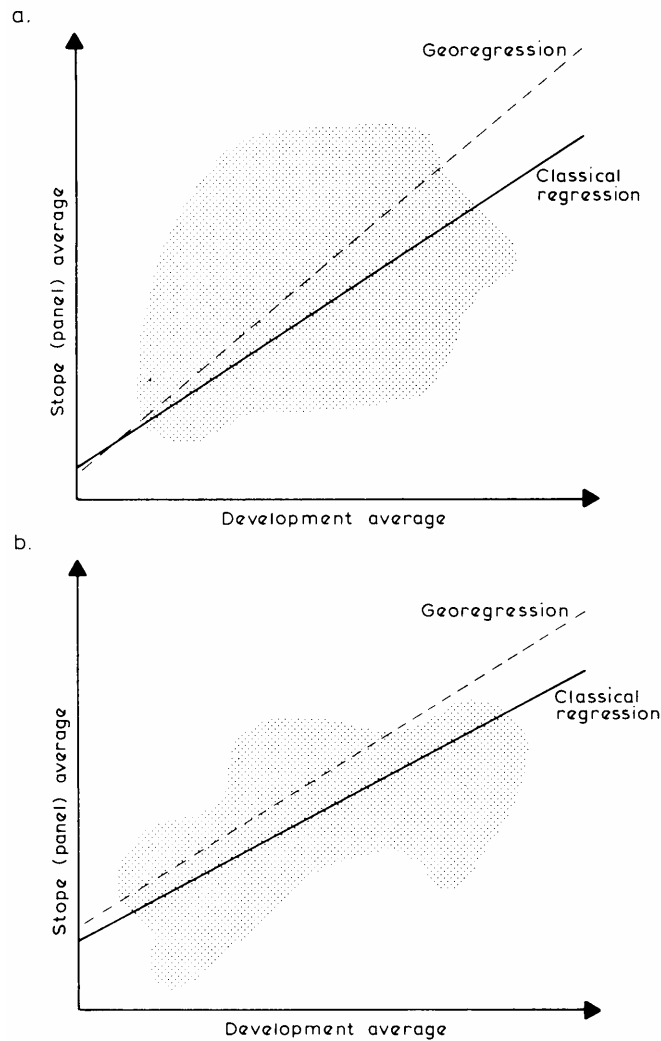


Fig. 5 Comparison of georegression and classical regression lines: (a) assay values; and (b) lode widths on Simms Lode (WOTF)

Standard errors can be calculated for georegression estimators, and it can be shown that in both cases illustrated in Fig. 5 the two lines lie within each other's 95% confidence intervals. The standard errors are calculated from the variances and covariances, which were derived from the semi-variogram. The estimation variance for the predicted value of a stopping panel is given by the equation:

$$\sigma_r^2 = s_1^2 - 2 \hat{b} s_{12} + (\hat{b} s_1^2)^2 + (\sigma_m(1 - \hat{b}))^2$$

σ_m was defined previously as the standard error associated with the estimate of the overall mean of the area being analysed. The formula given above holds for any straight line. For the supposedly optimal least-squares regression line, the expression may be reduced to

$$\sigma_{ls}^2 = s_1^2 - 2 \hat{b} s_{12} + (\sigma_m(1 - \hat{b}))^2$$

Weighted-average or kriging estimators

In the study described above the optimal-linear estimator used was the arithmetic mean of the sample values. The regression lines were constructed between this and the stope

average. In many cases the optimal-linear estimator will be significantly different from the arithmetic mean and will be a true-weighted mean, say T^* . The regression line may be constructed between T^* and the 'true' stope average to form a new estimator, $\hat{T}$, which would be

$$\hat{T} = \hat{a} + \hat{b}T^*$$

The difference in application between this and the special case of the arithmetic mean is that the calculated variances and covariances must take account of the different weights allocated to each sample. This is directly analogous to the difference between the calculation of the extension variance and the more general estimation variance in standard geostatistics. If a computer program is written to produce kriging estimates for stopes or blocks, the implementation of georegression adds about five statements to the kriging system and virtually no extra computing time. It should be noted, however, that the overall mean of the deposit (or zone) must be estimated prior to this. This will add to the total computing time, but if a large number of stopes is to be estimated, this extra cost will be small.

It will also be noticed that a simple regression line has been used rather than a polynomial or logarithmic form. As stated previously, a technique was desired that would be generally applicable. The fit of a complex curve depends too greatly on the characteristics of the particular data set, whereas straight lines have been used for all types of data in various past publications.

Three-dimensional deposits

Regression methods do not seem to have been applied in so-called three-dimensional deposits, perhaps because there was less opportunity to compare the estimated value of a block or stope with the actual local value obtained. In such mines and in open-pits much less 'stope' sampling is carried out, and there is less tendency to identify the ore as being from a particular site. On the other hand, the volume-variance relationship and disjunctive kriging have been applied almost exclusively to such deposits, so there would seem to be some potential for the use of georegression here. The application described below is on a deposit at a fairly early stage of evaluation - after diamond drilling - and should, therefore, be viewed in the light of current investigation.

Kvanefjeld uranium deposit

Geology and exploration

The Kvanefjeld plateau (Fig. 6) is situated 8 km NNE of the town of Narssaq, South Greenland. The plateau has an area of 2.5 km² and hosts a large syngenetic uranium deposit. The uranium, together with thorium, is disseminated in igneous rocks with massive outcrops of high natural radioactivity.¹⁵

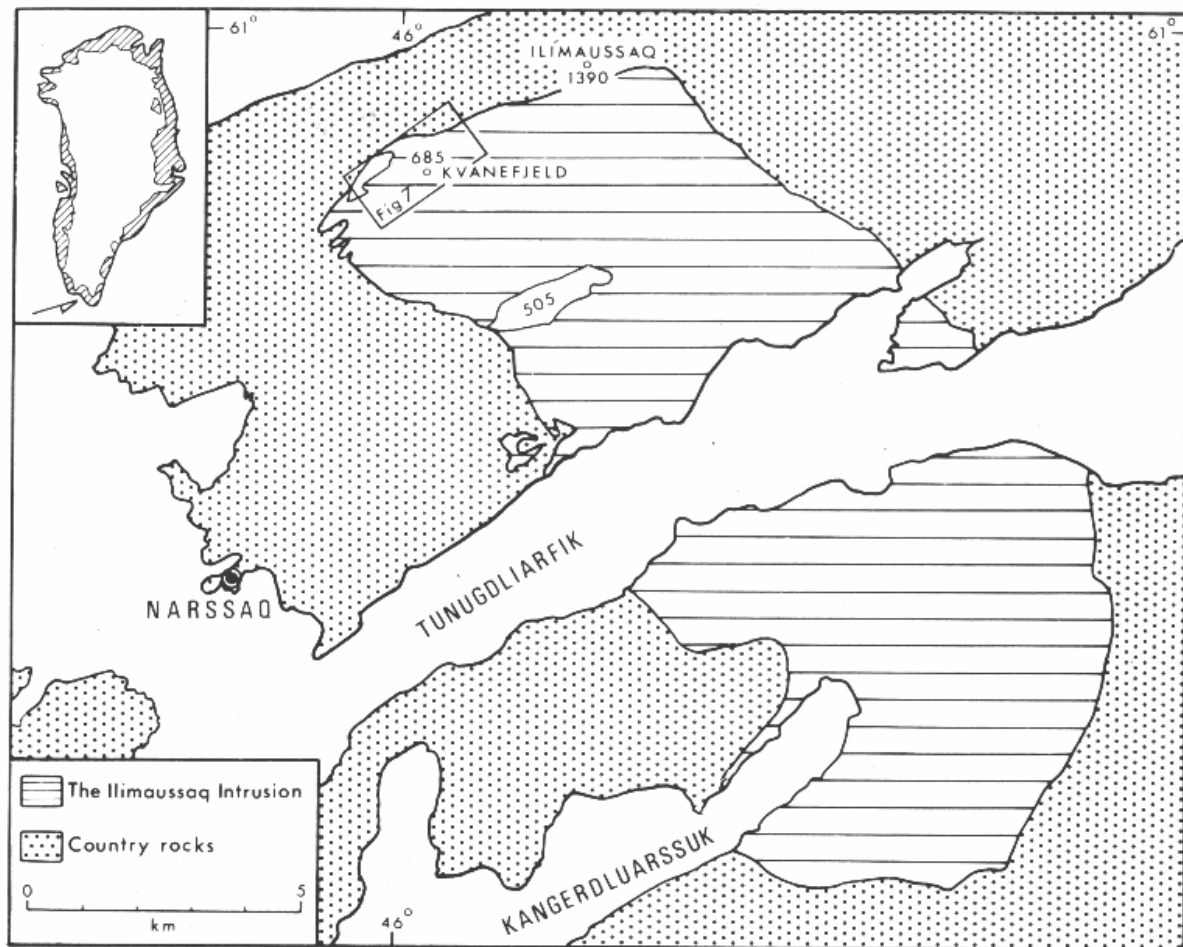


Fig. 6 Location map of Kvanefjeld uranium deposit and Ilímaussaq intrusion, South Greenland

The deposit under consideration is situated at the northern margin of the Ilímaussaq intrusive complex. This complex belongs to a number of alkaline intrusions that were emplaced into middle Proterozoic Ketilidian basement rocks during the Gardar period. 16 The Ilímaussaq intrusion is composed of alkaline and peralkaline syenites, nepheline syenites and a granite, and the geology and petrology have been thoroughly presented elsewhere. 17,18,19 The peralkaline nepheline syenites are comprised of a number of rock types that are unique with respect to mineralogical composition and structure. The occurrence of uranium is associated with the youngest nepheline syenite, the lujavrite, which is generally dark and finegrained with a pronounced magmatic lamination. 15,21 This rock type is enriched in both uranium and thorium as well as niobium, zirconium, beryllium, lithium, fluorine and rare earths. The radioactivity of the lujavrite is mainly controlled by disseminated crystals of steenstrupine, a uranium-thorium-bearing rare-earth phosphosilicate. 22,23

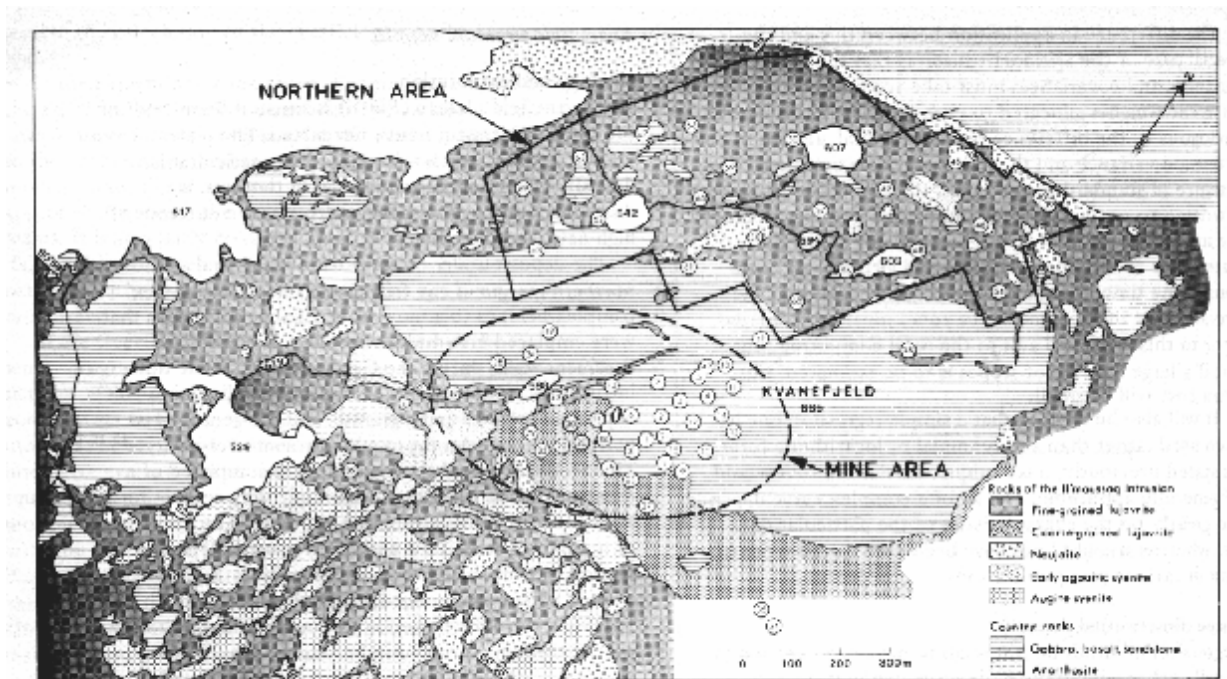


Fig. 7 Simplified geology of Kvanefjeld uranium deposit to show location of diamond drilling

Fig. 7 shows the whole Kvanefjeld area with simplified geology and the drilling pattern; the part of Kvanefjeld that is discussed in this paper is indicated as the 'mine area'. This name was given to the area because small amounts of ore have been mined south of drill-holes 23 and 26 for uranium-extraction experiments. The lujavrite has a surface exposure of 30-40%. Otherwise the area is characterized by a large number and variety of xenoliths - which include rocks derived from the roof of supracrustal lava and gabbro, as well as the earlier syenites of Ilímaussaq. These xenoliths are referred to as 'inclusions' in the discussion that follows. Inclusions generally have a low content of uranium, but some highly mineralized xenoliths are found, especially near the contact to the lujavrite.

The rock types within the mine area can be divided into four major groups. These are comprised of three lujavrite types: medium- to coarse-grained lujavrite (MC lujavrite), fine-grained black lujavrite and nauyasite lujavrite. The fourth group is comprised of all of the inclusions.

Exploration drilling and core assaying

Diamond drilling programmes have been carried out at Kvanefjeld in 1958, 1962, 1969 and 1977. Seventy holes were drilled with a total core length of 10 460 m. In the mine area 37 holes were available, all except three vertical. Each drill-core section was non-destructively analysed for potassium, uranium and thorium by γ -ray spectrometry, on a drill-core scanning device.²⁴ The uranium content obtained by this procedure gives the grade in adjacent core sections with an average length of 1 m. Neighbouring core sections generally overlap each other by a few centimetres, but this has been ignored for the purpose of the present analysis. Samples were labelled with a 'geology code' so that sorting of the data material by rock type was possible.

It should be noted that the part of the deposit that is considered in this study is comprised of ~4 the total reserves in Kvanefjeld. The total 'reasonably assured' reserves have been estimated by conventional methods to be 27 000 t uranium. Another 16 000 t uranium has

been classified as ‘additional reserves’. The average grade of the whole deposit has been estimated as 340 ppm uranium with a cutoff of 250 ppm uranium.^{20 24}

Uranium distribution

The histogram of 3 108 core sections available in the mine area is shown in Fig. 8; it is highly skewed and markedly bimodal. The first peak in the range 0-25 ppm uranium can be explained by looking at the individual rock-type histograms in Fig. 9. It is easily seen that very low-grade samples are generally found in ‘barren’ inclusions - which could possibly be selectively mined. A sample has been labelled as an inclusion if the content of xenolith is > 50%. Under this criterion a sample may contain up to 50% of mineralized lujavrite. The distribution of the inclusion values can, therefore, be viewed as a mixture of slightly mineralized xenolith and mineralized samples. The distribution of the uranium in the three different lujavrite types seems to be consistently of the log-normal type, although simple log-normal distributions do not fit the data. The difference between the distributions of uranium in the MC lujavrite and the two fine-grained lujavrites possibly reflect the activity of the magmatic as well as the post-magmatic processes that have taken place at Kvanefjeld.

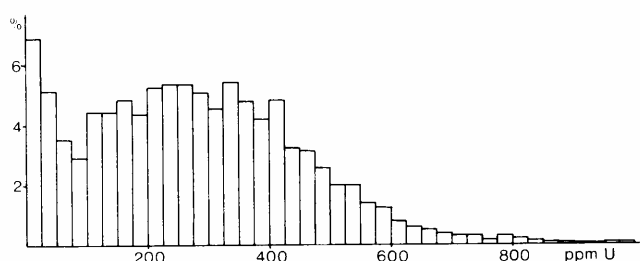


Fig. 8 Histogram of uranium content of drill-core sections in mine area, Kvanefjeld

Semi-variograms

Experimental semi-variograms were calculated along each drill-hole, ignoring rock type. These were then averaged to form an overall ‘vertical’ semi-variogram, which is shown in Fig. 10. No account was taken of possible proportional effect. Fig. 10 also shows the experimental semivariograms calculated within each lujavrite type. Even though these semi-variograms look very ‘noisy’, a distinct levelling off can be seen at a very short distance. This implies a short range of influence in each case. The graph also shows that no drift appears to be present. As would be intuitively expected, the ‘single-rock type’ semivariograms have lower experimental values than the overall one. This is because the material within one geological unit is more homogeneous, and, hence, it will have a lower variance of values. Attempts to calculate horizontal semivariograms were frustrated by the large and irregular spacing of the drill holes.

Only the overall semi-variogram was used in this study, because of the difficulties in fitting stable models to the individual geology types. It can be seen from Fig. 10 that the overall semi-variogram has a high nugget effect, at $5.6 \times 10^3 \text{ ppm}^2$. There appears to be an intermediate sill at $22 \times 10^3 \text{ ppm}^2$ and a final sill at $31 \times 10^3 \text{ ppm}^2$. In consequence the semi-variogram was expected to be comprised of two spherical components and the nugget effect.

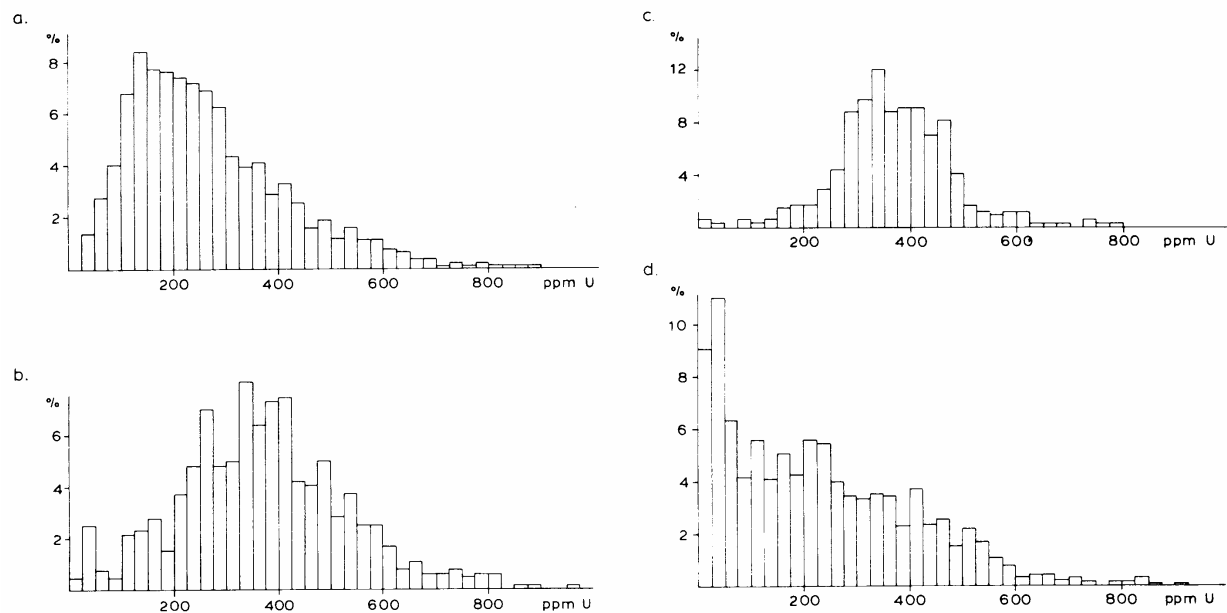


Fig. 9 Histograms of uranium content of drill-core sections sorted by 'geology code'; (a) MC lujavrite; (b) fine-grained black lujavrite; (c) fine-grained naujakasite lujavrite; and (d) xenoliths (inclusions)

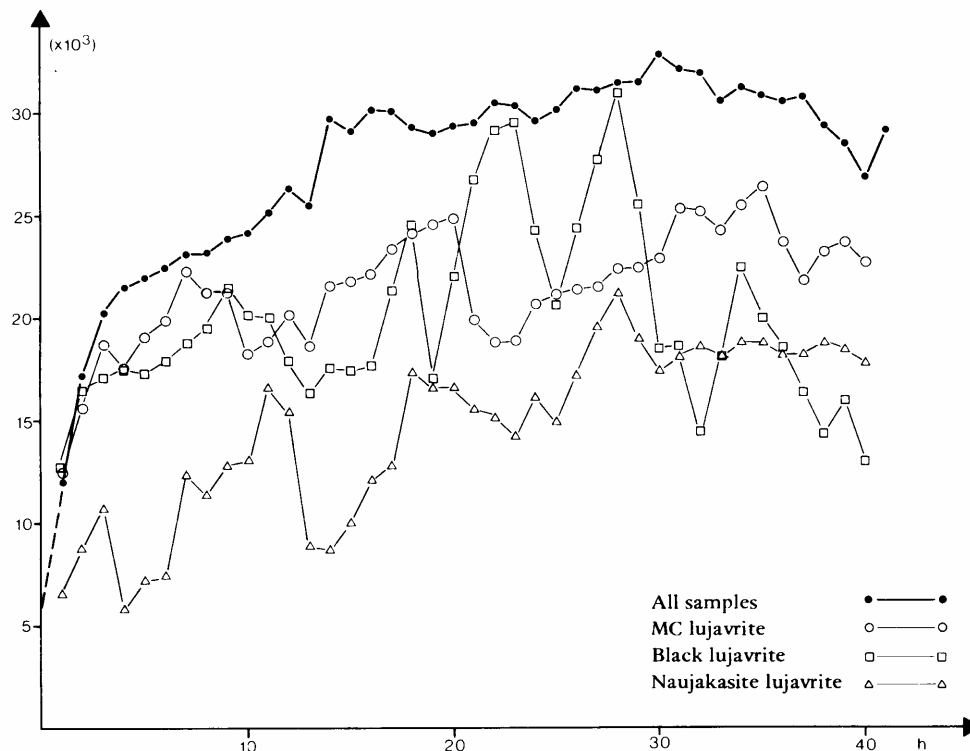


Fig. 10 Experimental semi-variograms for diamond drill holes in Kvanefjeld mine area

Trial and error were used to estimate the parameters of this model, and a first approximation was given by the following: $C_0 = 5600 \text{ ppm}^2$; $a_{1b} = 3.5 \text{ m}$; $C_{1b} = 13\,600 \text{ ppm}^2$; $a_{2b} = 30.0 \text{ m}$ and $C_{2b} = 11\,800 \text{ ppm}^2$. The subscript b has been used to emphasize that this model is only an approximation, since the 'samples' that went into the experimental semi-variogram were actually core sections of length 1 m. From this first approximation,²⁵ a model for 'point' samples can be constructed. The regularized curve given by this model for samples 1-m in length can then be computed and compared with the experimental semi-variogram. From this process, the 'point' model was found to have two ranges of influence

of 2.5 and 29 m, respectively, sills of 16 930 and 12 000 ppm², respectively, and a nugget effect of 5600 ppm². This model implies that samples within 2.5 m are highly spatially correlated, but with ~25% of the total variation being random. Samples within 29 m are also correlated, but to a much weaker extent, since the random component is now ~69%. This factor is clearly seen from the difference between the point model and the regularized one shown in Fig. 11.

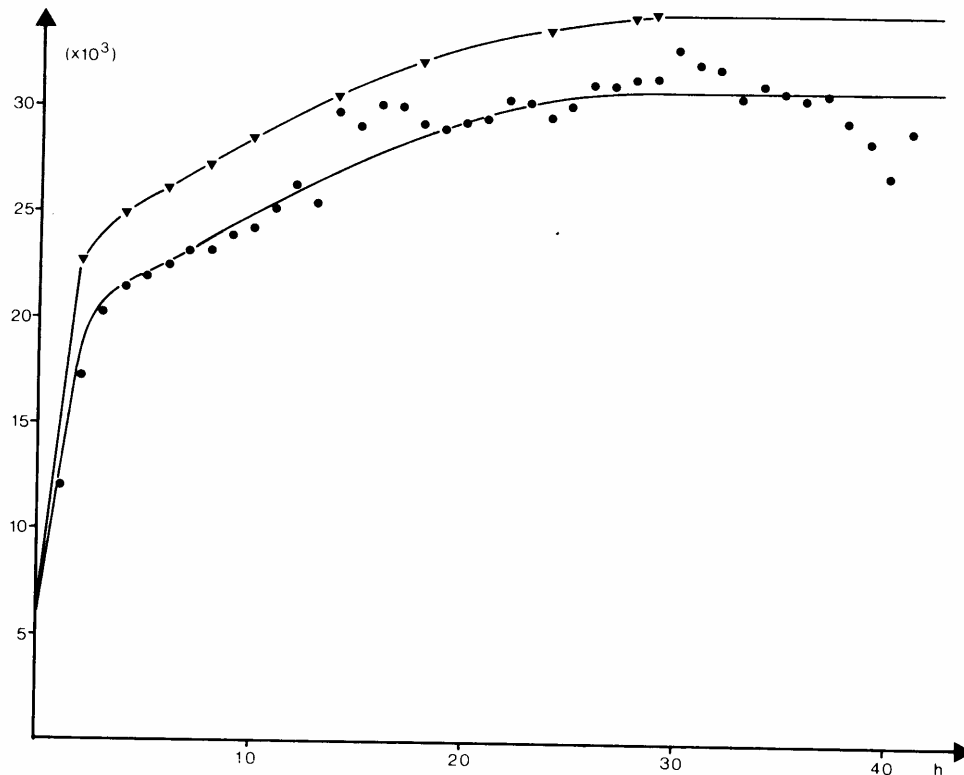


Fig. 11 Semi-variogram models for diamond drill holes in mine area. All samples

Estimation

In the estimation of any volume of ground from a given sample set two immediate problems arise: the size and shape of the blocks; and the orientation of the block pattern with respect to the drill-holes. At a later stage in development, when a mine plan is being drawn up and decisions have been made upon the mining method and so on, these questions will be settled automatically. At this stage, however, the placement of the blocks and their size can have significant effects upon the estimate of the global ore reserve. Generally, large blocks that are intersected by many boreholes will yield small estimation variances, i.e. give more reliable figures. On the other hand, if large blocks are chosen, only poor selection between 'ore' and 'waste' is possible. Since this deposit is known to contain a large quantity of barren inclusions, a smaller block would be more desirable. The estimates of the values of small blocks will, however, have much larger estimation variances, eventually reaching a level where the estimate is totally unreliable.

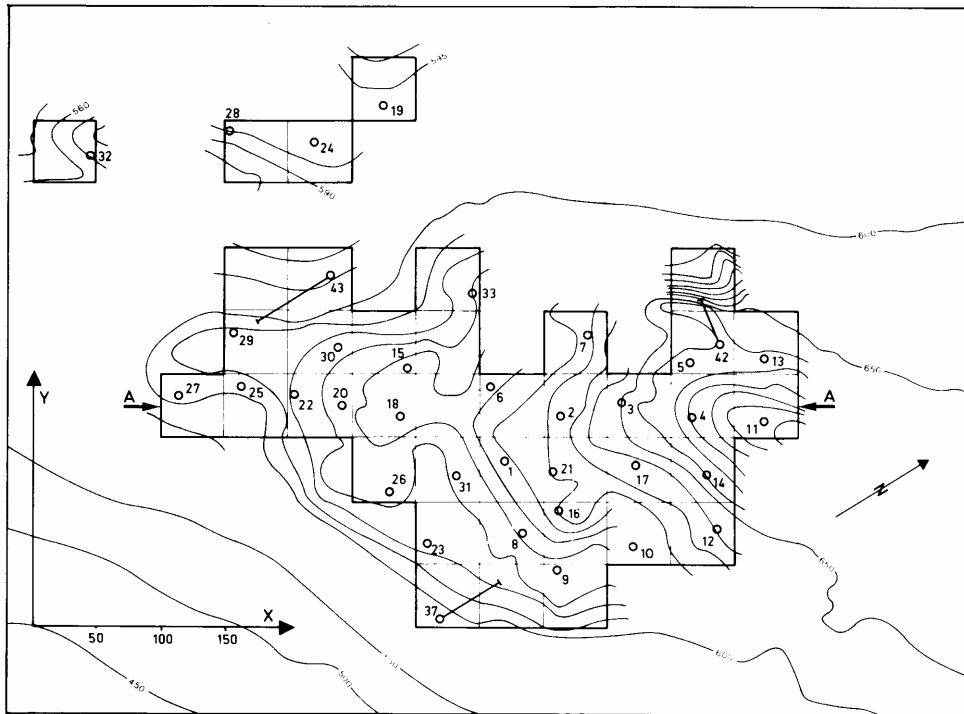


Fig. 12 Random stratified grid (RSG) fitted to drill-hole pattern in mine area

In deference to these considerations it was decided to fit a random stratified grid (RSG) to the relatively sparse drilling pattern.²⁶ The advantage of the RSG is mainly that estimation is more efficient, since equal-size blocks are used, and because each block should be estimated with about the same amount of error. The first estimate for the 'size' of the RSG is made by the division of the area of interest by the number of drill-holes. If the distribution of drill-holes seems to be uniform in space, a square RSG will probably give a good coverage. In the Kvanefjeld mine area it was found that a 50 m x 50 m RSG gives a remarkably good fit to the spatial distribution of the boreholes. Most grid squares contain one hole, two contain no holes and two contain two holes each. The two grid squares with no drill-hole intersection were included because they appeared to be surrounded by mineralized areas, as may be seen in Fig. 12. This gives a total of 39 blocks to be estimated on each bench level.

Block kriging

Having chosen a block size of 50 m in the 'horizontal' direction, and the orientation of the blocks, it remained to choose a bench height for the investigation. Since samples consist of core sections 1 m long, there is a lower limit for the possible bench height. Very thin benches do not, however, make much sense in conjunction with a 50-m block. The thicknesses of the xenoliths range from 2 to 10 m, and these seem to occur as lenticular bodies. With this in mind, a bench height of 10 m was selected. Kriging was carried out by the FORTRAN IV computer program TREREG. This is a version of the commercial package GSTOKOS, which includes georegression calculations. The program will perform both least-squares and perpendicular-distance georegression as requested by the user. The kriging procedure that was used to evaluate the block estimates was carried out in three dimensions: that is, not only were samples within the bench considered, but also samples on benches above and below the block. As with all such computer programs, a search volume

has to be defined by the user, within which the program should search for samples to include in the kriging procedure. Because of the set-up of TREREG (and GSTOKOS) this was specified in terms of the number of blocks to each side of, and above and below, the block that was to be estimated.

Consideration was given to the model of the semi-variogram, which gave the largest range of influence as 29 m; the search area was chosen to include one block on either side and three above and below the current block. This gave a total search area of 150 m x 150 m x 70 m. It is clear that samples found on the edges of this area will be outside the range of influence of the block; however, these samples will automatically receive little weight in the kriging system so no special precautions need be taken. The problems and techniques of kriging in three dimensions and the special routines used in the computer programs have been described elsewhere.

Global estimates

Three different global estimates were calculated in the mine area and kriging estimates were produced for each block and have been summarized. In addition, these kriging estimates have been 'corrected' (a) by least-squares georegression and (b) by perpendicular-distance georegression. Fig. 13 shows a vertical section through the deposit (line A-A, Fig. 12), and the pattern of blocks to be estimated. Drill-holes within the line of blocks are shown as solid lines, and those off the line, but within the search area, as broken lines. As can be seen, the computer program may estimate blocks up into the air or into the barren roof rocks. These are eliminated manually. It can also be seen that many blocks may be estimated at the 'bottom' of the deposit, with little information. These blocks will, of course, have very large estimation variances.

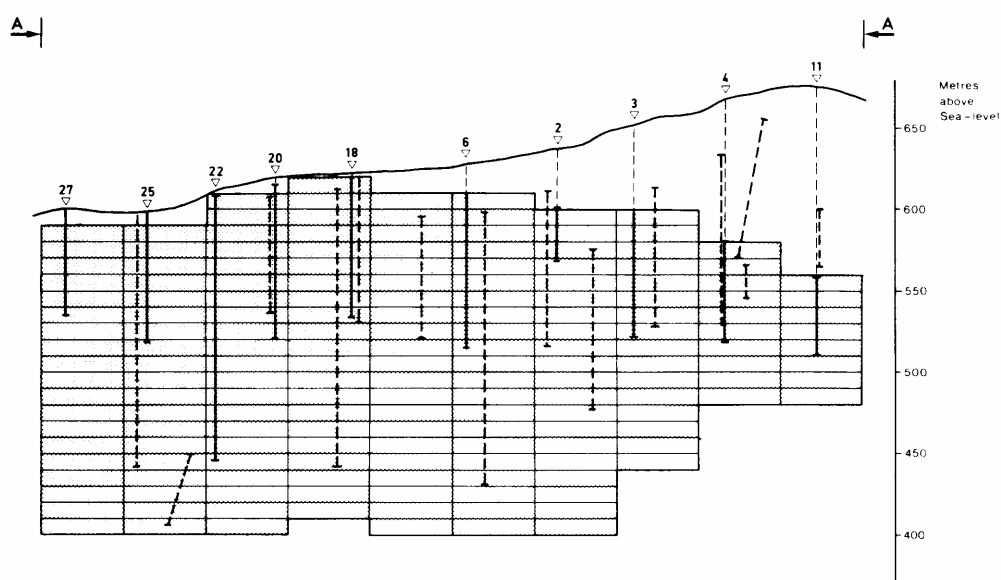


Fig. 13 Typical vertical section through block plan in mine area. Section line is A-A in Fig. 12

For comparison purposes a cutoff grade of 300 ppm uranium was chosen. This is probably rather higher than would be chosen in practice. Grade and tonnage calculations in the Kvanefjeld mine area have, however, previously been published¹⁵ with the use of this figure. The estimate cited was based on the classical method of triangles as illustrated in

Fig. 14. The block pattern shown involves quite a different volume from the RSG, which would *a priori* lead to a different estimate. In fact this is not the case.

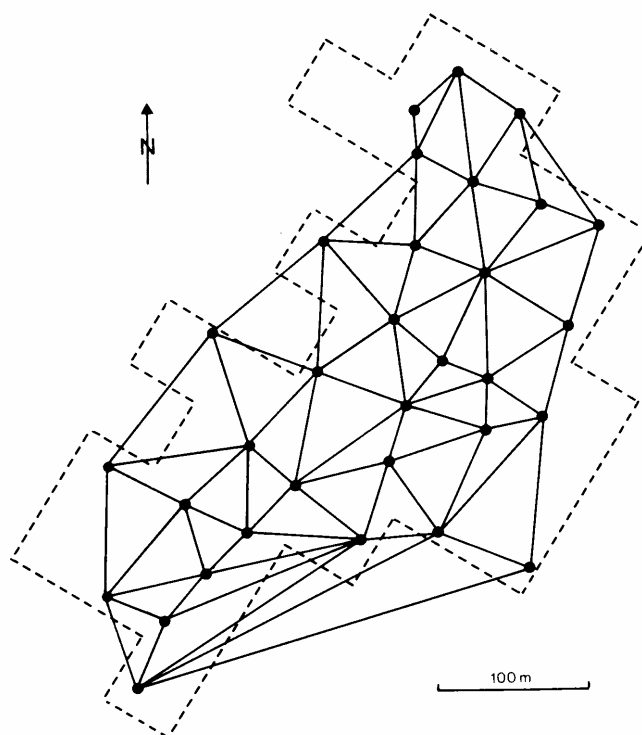


Fig. 14 Triangles used in conventional estimation of global reserve in mine area

Table 1 shows the various estimates of grade and tonnage in the mine area given a cutoff grade of 300 ppm. A constant density of 2.7 t/m^3 was used in all of these calculations. Surprisingly, there is little difference between the kriging estimate and that found by the triangular method. The two georegression estimates, on the contrary, give much lower tonnage estimates, although the average grades do not differ so significantly. A fuller comparison of the three geostatistical estimates can be seen in Fig. 15. These histograms may also be compared with the original histogram of the core sections in Fig. 8. The enormous smoothing effect of the georegression when applied to the kriged block estimates becomes immediately apparent.

Table 1 Comparison of various methods of estimation

	Kriging	Kriging plus least squares	Kriging plus perpendicular distance	Sorensen et al ¹⁵
Ore tonnage 10^3 t	15 862	6345	7695	18 568
Uranium tonnage, t	5270	1965	2543	5760
Mean grade, ppm U	332	310	326	310

Cutoff, 300 ppm.

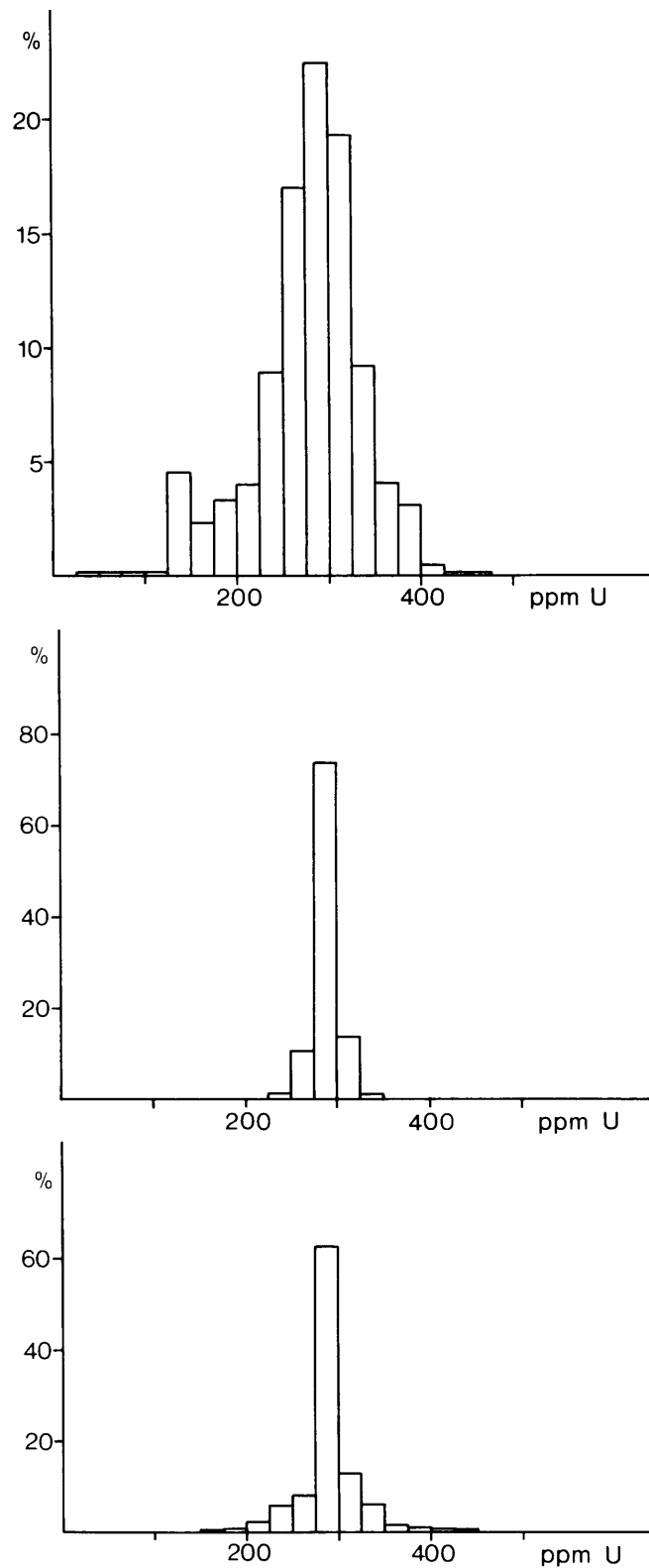


Fig. 15 Histograms of block estimates produced in mine area by (a) kriging; (b) least-squares georegression; and (c) perpendicular-distance georegression

The least-squares histogram is obviously totally unrealistic, since almost all of the blocks have been allocated the average grade of the deposit. Royle²⁸ has shown that many estimators are highly influenced by the nugget effect, and that tends to emphasize the bias

on the grade-tonnage relationship. Many authors (such as Marchal³¹) have shown that even kriging, which produces the best linear unbiased estimate for each individual value, produces a bias on the estimated grade-tonnage curve. Since the Kvanefjeld uranium has a very poor spatial structure with a high nugget effect, the difference between a (biased) kriging estimator and an (unbiased) georegression one is very great. The individual kriged block estimates are adjusted more or less toward the mean of the deposit, giving a variance much closer to that theoretically expected from such large blocks. If smaller blocks could be estimated, the adjustment would not be so severe.

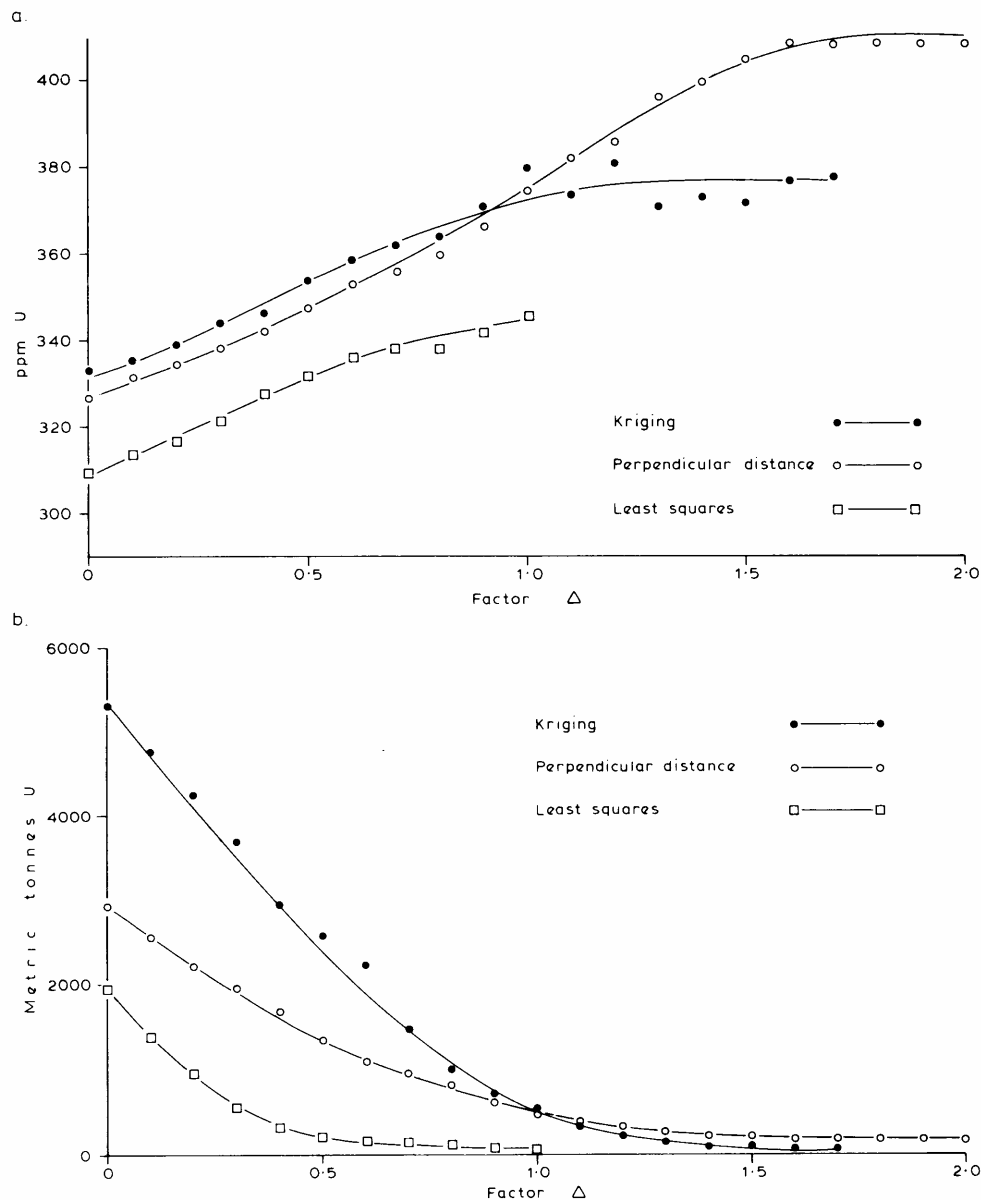


Fig. 16 Estimated global reserves in mine area against 'confidence level': (a) average grade above cutoff; and (b) tonnage above cutoff (specific gravity 2.7)

Reliability of estimates

The estimates of grade and tonnage given in Table 1 take no account of the estimation error on each block value. If it were assumed that the errors followed a normal distribution, it would be possible to obtain a (say) 95% confidence interval around the estimate by, respectively, adding and subtracting two estimation standard deviations (standard errors) from the estimate: that is, if twice the standard error is subtracted from a given block estimate, a lower 97.5% confidence limit would be obtained. The user can be 'almost sure' that the true block value lies above this limit. If this lower confidence limit is above the specified cutoff grade, the user can be 'almost sure' that the average grade of the block is above cutoff.

In the Kvanefjeld mine area, no block can be said to be above a cutoff of 300 ppm if the 'two standard error' criterion is applied to the kriging block estimates. From the perpendicular-distance georegression estimates, however, six blocks out of 640 satisfy the criterion. This process may be repeated for different (presumably lower) levels of confidence by the evaluation of the number of blocks that pass the criterion:

$$\text{Estimated grade} - \Delta \times \text{standard error} > \text{cutoff}$$

Fig. 16 shows two graphs of 'confidence level', Δ , against (a) average grade above cutoff and (b) tonnage of ore above cutoff. Once again the effect of the volume-variance relationship corrections can be clearly seen. Fig. 17 shows how the standard errors vary when kriging alone is used: they range from 40 ppm to 180 ppm. Kriging with perpendicular-distance georegression produces standard errors that vary little, in the range 40-44 ppm.

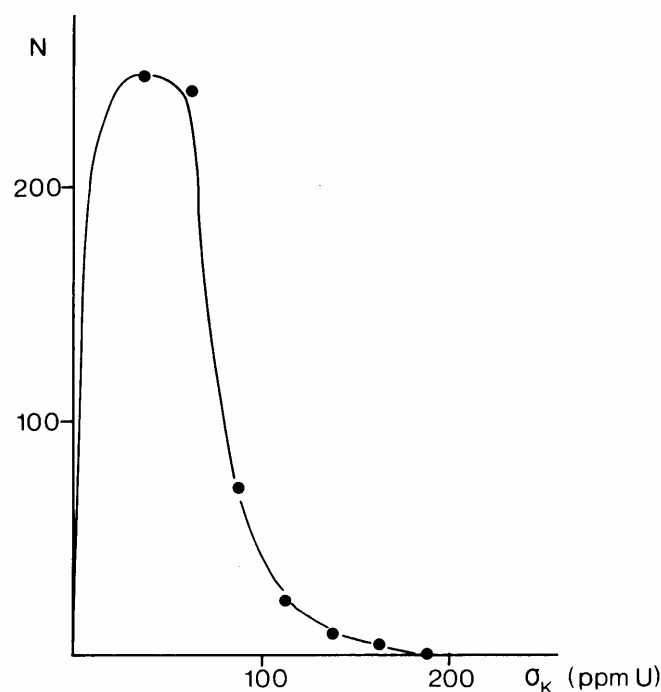


Fig. 17 Number of kriged blocks against kriging standard error, σ_K

Conclusion

The problem of bias on the grade-tonnage curve produced by the consideration of the linear estimators has been variously described in the literature as the 'regression' or the 'volume-variance' effect. A solution to this problem is proposed that is simple and apparently generally applicable. At the very least, it may fill the gap until disjunctive kriging reaches a wider audience. Two case studies have been discussed here. In one the georegression relationship provided a very close approximation to an empirical regression line. Since the latter was based on 2500 additional stope samples, it is felt that this was a fairly rigorous test. In the second (three-dimensional) case study, the georegression approach has been shown to produce more realistic results at an early stage of analysis. Both of the applications described in this paper concerned highly erratic types of mineralization, i.e. high nugget effects. In consequence of this, it appears that the perpendicular-distance georegression is the most suitable in both cases. Some further work needs to be carried out, to determine the criterion 'large nugget effect', since this obviously influences the choice between least-squares and perpendicular-distance georegressions.

It should also be emphasized that the georegression method. The two georegression estimates, on the contrary, give much lower tonnage estimates, although the average grades do not differ so significantly. A fuller comparison of the three geostatistical estimates can be seen in Fig. 15. These histograms may also be compared with the original histogram of the core sections in Fig. 8. The enormous smoothing effect of the georegression when applied to the kriged block estimates becomes immediately apparent.

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criterion. This process may be repeated for different (presumably lower) levels of confidence by the evaluation of the number of blocks that pass the criterion:

$$\text{Estimated grade} - \Delta \times \text{standard error} > \text{cutoff}$$

approach may be used with any linear estimator including inverse distance - provided that a semi-variogram model can be derived from the sample data.

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